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ABSTRACT. *A new family of methods, the so-called two-parameter* vector penalty-projection *(VPP$_{r,\varepsilon}$) methods, is proposed where an original* penalty-correction step *for the velocity replaces the standard scalar pressure-correction one to calculate flows with divergence-free velocity. This allows us to impose the desired boundary condition to the end-of-step velocity-pressure variables without any trouble. The counterpart to pay back is that in these methods, the constraint on the discrete divergence of velocity is only satisfied approximately as $\mathcal{O}(\varepsilon\delta t)$ within a* penalty-correction step *and the penalty parameter $0 < \varepsilon \leq 1$ must be decreased until the resulting splitting error is made negligible compared to the time discretization error. However, the crucial issue is that the linear system associated with the projection step can be solved all the more easily as $\varepsilon\delta t$ is smaller. Finally, the* vector penalty-projection method *(VPP$_{r,\varepsilon}$) has several nice advantages: the Dirichlet or open boundary conditions are not spoiled through a scalar pressure-correction step. Moreover, this method can be generalized in a natural way for variable density or viscosity flows and we show that the vector correction step can be made quasi-independent on the density or viscosity variables (and also on the non-linear terms) if $\eta = \varepsilon\delta t$ is taken sufficiently small. These terms can be then neglected in practical schemes.*

KEYWORDS: *Vector penalty-projection methods, projection methods, artificial compressibility, Navier-Stokes equations, incompressible and variable density flows, cell-centered MAC scheme.*

1. Introduction: overview of methods dealing with the divergence constraint

Let us consider the unsteady Navier-Stokes problem in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leq 3$) with Dirichlet boundary conditions $\mathbf{v}_{|\Gamma} = \mathbf{v}_D$ on $\Gamma = \partial\Omega$ and $\mathbf{f}$, q (a volumic mass source), Re (Reynolds number), an initial condition $\mathbf{v}(t\!=\!0) = \mathbf{v}_0$ given:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\mathbf{v} + \boldsymbol{\nabla} p = \mathbf{f} \quad \text{with} \quad \boldsymbol{\nabla}\cdot\mathbf{v} = q \quad \text{in } \Omega \times (0,T). \qquad [1]$$

1.1. *Fully-coupled Navier-Stokes solvers*

With usual notations for the semi-discretization in time, the *linearly implicit Euler method* with $\bar{\mathbf{v}}^0 = \mathbf{v}_0$ and a given time step $\delta t > 0$ reads: for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$, find $\bar{\mathbf{v}}^{n+1}$ and $\bar{p}^{n+1}$ satisfying $\bar{\mathbf{v}}^{n+1}_{|\Gamma} = \mathbf{v}^{n+1}_D$ such that:

$$\frac{\bar{\mathbf{v}}^{n+1} - \bar{\mathbf{v}}^n}{\delta t} + (\bar{\mathbf{v}}^n \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\bar{\mathbf{v}}^{n+1} + \boldsymbol{\nabla}\bar{p}^{n+1} = \mathbf{f}^{n+1} \qquad \text{in } \Omega \quad [2]$$

$$\boldsymbol{\nabla}\cdot \bar{\mathbf{v}}^{n+1} = q^{n+1} \qquad \text{in } \Omega \quad [3]$$

Here $\bar{\mathbf{v}}^n, \bar{p}^n$ are desired to be first-order approximations of the continuous velocity and pressure solutions $\mathbf{v}(t_n), p(t_n)$ at time $t_n = n\delta t$. Of course, higher-order schemes are generally used for practical computations. For the sake of simplicity in the numerical procedure, a semi-implicit scheme where the non-linear term is linearized on the first term is often chosen since it does not suffer from stability conditions. The error analysis of the fully implicit scheme is simpler but it also makes it necessary to use a quasi-Newton algorithm to solve the corresponding non-linear system at each time step in the practical calculations. It is well-known that such a *fully-coupled scheme*, with any stable space discretization, yields an ill-conditionned and indefinite algebraic system at each time step, especially for small spatial mesh steps h. The design of efficient fully-coupled solvers remains a general problem still largely open; see e.g. [CIH 99].

1.2. *Augmented Lagrangian and artificial compressibility methods*

Falling into the class of *semi-coupled* solvers where the velocity components are still gathered, the *augmented Lagrangian method* which is classical in optimization [FOR 83], with the parameters $0 < \lambda < 2r + 2/\mathrm{Re}$ and the initial condition p^0 given, reads:

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\mathbf{v}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\mathbf{v}^{n+1} - r\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \mathbf{v}^{n+1} - q^{n+1}\right) + \boldsymbol{\nabla}p^n = \mathbf{f}^{n+1}$$

$$p^{n+1} = p^n - \lambda\left(\boldsymbol{\nabla}\cdot \mathbf{v}^{n+1} - q^{n+1}\right).$$

This algorithm must in fact be iterated at each time step (not given here for sake of shortness) to produce good approximations of the solution to the reference method, i.e. the coupled implicit Euler scheme [2]-[3] for unsteady flows, see [KHA 00] for details. However, the convergence to a "divergence free" velocity, varying like $\mathcal{O}(1/r)$, is all the faster as the augmentation parameter $r \gg 1$ is larger but the resulting system becomes here also ill-conditioned which greatly increases the solution cost, especially for 3-D problems. On the contrary, the *standard Uzawa* algorithm with $r = 0$ converges very slowly or may not converge at all for strongly non-linear problems. With the choice $\lambda = r = \delta t/\varepsilon$, the method also yields the *implicit artificial (or pseudo) compressibility* method studied by Chorin (1967) and Temam (1968) [TEM 86]. Indeed, the scheme then corresponds to the implicit first-order discretization of the following singularly perturbed continuous problem where $\mathbf{v}_\varepsilon(0) = \mathbf{v}_0$ and also $p_\varepsilon(0)$ are given:

$$\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \boldsymbol{\nabla})\mathbf{v}_\varepsilon - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\mathbf{v}_\varepsilon + \boldsymbol{\nabla} p_\varepsilon = \mathbf{f} \qquad \text{in } \Omega \times (0,T), \qquad [4]$$

$$\varepsilon\, \partial_t p_\varepsilon + \boldsymbol{\nabla}\cdot \mathbf{v}_\varepsilon = q \qquad \text{in } \Omega \times (0,T), \qquad [5]$$

with $\mathbf{v}_\varepsilon = \mathbf{v}_D$ on Γ. This *artificial compressibility problem* (for $q = 0$) can be proved to converge in some sense to the Navier-Stokes system when ε tends to zero, see [TEM 86] and [ANG 08] for new convergence results like Theorem 2.1.

1.3. *Scalar projection and penalty-projection methods*

With a completely different approach, the class of fractional-step or splitting methods, namely the *projection methods* originally introduced by Chorin (1968) and Temam (1969) [TEM 86], became very popular because they are much cheaper. They produced an important literature with many variants of the type of pressure-correction or velocity-correction schemes and we refer to the complete review of Guermond *et al.* [GUE 06] and the references therein. Their principle lies in the Helmholtz-Hodge orthogonal decomposition of the space $L^2(\Omega)^d = \mathbf{H} \oplus \mathbf{H}^\perp$ where, see e.g. [TEM 86]: $\mathbf{H} = \{\mathbf{u} \in L^2(\Omega)^d,\ \boldsymbol{\nabla}\cdot \mathbf{u} = 0,\ \mathbf{u}\cdot \mathbf{n} = 0 \text{ on } \Gamma\}$ and $\mathbf{H}^\perp = \{\boldsymbol{\nabla}\phi,\ \phi \in H^1(\Omega)\}$.
This allows for a predicted velocity field to be corrected with a "pressure" gradient by the Leray orthogonal projection $P_{\mathbf{H}}$ onto the space of solenoidal fields $\mathbf{H}$. As an example, let us consider the so-called *penalty-projection* method recently proposed and numerically investigated by Jobelin *et al.* [JOB 06, JOB 07], also theoretically analysed in [ANG 06] – the first one being due to Shen (1992). For $r_1 \geq 0$ and p^0 given, the predicted velocity $\tilde{\mathbf{v}}^{n+1}$ satisfies in Ω:

$$\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\tilde{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\tilde{\mathbf{v}}^{n+1} - r_1 \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right) + \boldsymbol{\nabla} p^n = \mathbf{f}^{n+1}, \qquad [6]$$

with $\tilde{\mathbf{v}}^{n+1} = \mathbf{v}_D^{n+1}$ on Γ. This system condition number varies as $\mathcal{O}(r_1)$ [FEV 08].
Then the pressure correction step reads for $r_2 \geq 0$:

$$-\boldsymbol{\Delta}\phi = \frac{1}{\delta t}\left(q^{n+1} - \boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1}\right) \ \text{ in } \Omega \quad \text{ with } \quad \boldsymbol{\nabla}\phi \cdot \mathbf{n} = 0 \ \text{ on } \Gamma \quad [7]$$

$$\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1} + \delta t \boldsymbol{\nabla}\phi = 0 \ \text{ and } \ p^{n+1} = p^n + \phi - r_2\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right) \ \text{ in } \Omega. \quad [8]$$

Under this general form, the standard *incremental projection* method from Goda (1979) is recovered with $r_1 = r_2 = 0$ which makes it very simple and attractive. However, the usual projection methods suffer from some drawbacks:

1) First, the original Dirichlet boundary condition on the velocity degenerates into an inconsistent Neumann boundary condition for the pressure increment ϕ due to the *scalar pressure correction* step and so the end-of-step velocity only satisfies: $\mathbf{v}^{n+1}\cdot \mathbf{n} = \tilde{\mathbf{v}}^{n+1}\cdot \mathbf{n} = \mathbf{v}_D^{n+1}\cdot \mathbf{n}$. This creates artificial pressure boundary layers

which spoil the pressure numerical solution. They can be reduced with the *rotational projection* method from Timmermans *et al.* (1996) corresponding to the previous algorithm with $r_1 = 0, r_2 = 1/\mathrm{Re}$ [GUE 04] or with the standard penalty-projection ($r_1 = r_2 = r$) or the rotational version ($r_1 = r, r_2 = r + 1/\mathrm{Re}$) for which they are nearly suppressed with moderate values of r between 1 and 10 [JOB 06, FEV 08].

2) When an outflow boundary condition of the "Neumann" type like in Section 3 is imposed on a part of the boundary, it degenerates into a homogeneous Dirichlet condition for ϕ and the convergence of the standard projection methods is quite poor. Indeed, the $l^2(L^2)$ norm of the velocity and pressure errors converges as $\mathcal{O}(\delta t)$ and $\mathcal{O}(\delta t^{\frac{1}{2}})$, respectively for a second-order time scheme, see [GUE 06]. This is clearly improved with the rotational projection [GUE 06] or with the penalty-projection methods if r takes moderate values, typically $1 \leq r \leq 10$, see [JOB 06, FEV 08].

3) In the case of variable density flows, as multiphase flows, the scalar correction step is naturally modified depending now on the density ϱ^{n+1} [GUE 00], as follows:

$$-\boldsymbol{\nabla}\cdot\left(\frac{\delta t}{\varrho^{n+1}}\boldsymbol{\nabla}\phi\right) = q^{n+1} - \boldsymbol{\nabla}\cdot\tilde{\mathbf{v}}^{n+1} \ \text{ in } \Omega \quad \text{ with } \quad \boldsymbol{\nabla}\phi\cdot\mathbf{n} = 0 \ \text{ on } \Gamma$$

$$\varrho^{n+1}\left(\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1}\right) + \delta t\boldsymbol{\nabla}\phi = 0, \quad p^{n+1} = p^n + \phi - r_2\left(\boldsymbol{\nabla}\cdot\tilde{\mathbf{v}}^{n+1} - q^{n+1}\right) \ \text{ in } \Omega.$$

Thus, it introduces an "artificial" dependence on ϱ^{n+1} of the correction of $\tilde{\mathbf{v}}^{n+1}$: in the case of large density ratios, it generally prevents us from solving ϕ quasi-exactly which should be to satisfy the divergence constraint, whereas a standard precision in agreement with the discretization errors would be sufficient to get only the pressure!

4) The last drawback, less traditional, addresses the fictitious domain methods using a penalization to impose an immersed boundary condition; see [ANG 99, KHA 00] and the references therein. Here, the L^2-volumic penalty inside a moving subdomain $\omega(t) \subset \Omega$ with $b(t) = \chi_{\omega(t)}/\eta$, $\eta > 0$, modifies the projection method as below:

$$\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n\cdot\boldsymbol{\nabla})\tilde{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\tilde{\mathbf{v}}^{n+1} + \boldsymbol{\nabla}p^n + b^{n+1}\left(\tilde{\mathbf{v}}^{n+1} - \mathbf{v}_D^{n+1}\right) = \mathbf{f}^{n+1}$$

$$-\boldsymbol{\nabla}\cdot\left(\frac{\delta t}{1 + \delta t\, b^{n+1}}\boldsymbol{\nabla}\phi\right) = q^{n+1} - \boldsymbol{\nabla}\cdot\tilde{\mathbf{v}}^{n+1} \ \text{ in } \Omega \quad \text{ with } \quad \boldsymbol{\nabla}\phi\cdot\mathbf{n} = 0 \ \text{ on } \Gamma$$

$$\left(\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1}\right) + \delta t\boldsymbol{\nabla}\phi + \delta t\, b^{n+1}\left(\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1}\right) = 0, \quad p^{n+1} = p^n + \phi \ \text{ in } \Omega.$$

Thus, it yields a H^1-penalty for ϕ with a diffusion coefficient varying as $\mathcal{O}(\eta)$ when $\eta \to 0$, i.e. a singularly perturbed problem more difficult to treat numerically. This also gives an additional insight to our first point about the pressure boundary layers.

2. The vector penalty-projection (VPP$_{r,\varepsilon}$) methods for incompressible flows

The *vector penalty-projection* methods are derived to overcome most of the latter drawbacks with a compromise between the best properties of both classes: the augmented Lagrangian (without iterations) and splitting methods under a vector form, i.e. seeking a vector correction $\hat{\mathbf{v}}$ to inherently respect the velocity boundary conditions.

2.1. *The two-step artificial compressibility (VPP$_\varepsilon$) method*

A first two-step VPP scheme with a penalty parameter $0 < \varepsilon \leq \delta t$ is devised, for $\mathbf{v}^0 \in H^1(\Omega)^d$, $p^0 \in L^2_0(\Omega)$ given, as follows: $\forall n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$,

$$\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\tilde{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\tilde{\mathbf{v}}^{n+1} + \boldsymbol{\nabla} p^n = \mathbf{f}^{n+1}$$

$$\frac{\varepsilon}{\delta t}\left(\frac{\hat{\mathbf{v}}^{n+1}}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\hat{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\hat{\mathbf{v}}^{n+1}\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \hat{\mathbf{v}}^{n+1}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right)$$

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}}^{n+1} + \hat{\mathbf{v}}^{n+1}, \quad \text{and} \quad p^{n+1} = p^n - \frac{\delta t}{\varepsilon}\left(\boldsymbol{\nabla}\cdot \mathbf{v}^{n+1} - q^{n+1}\right)$$

with $\tilde{\mathbf{v}}^{n+1}_{|\Gamma} = \mathbf{v}^{n+1}_D$ and $\hat{\mathbf{v}}^{n+1}_{|\Gamma} = 0$ to satisfy the boundary condition on $\mathbf{v}$. Summing the two steps yields the implicit artificial compressibility scheme written in section 1.2.

2.2. *The two-parameter vector-penalty projection (VPP$_{r,\varepsilon}$) methods*

The two-parameter vector penalty-projection methods with an augmentation parameter $r \geq 0$ and a penalty parameter $0 < \varepsilon \leq 1$ are then devised. They read for $\mathbf{v}^0 \in H^1(\Omega)^d$ and $p^0 \in L^2_0(\Omega)$ given: for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$,

$$\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\tilde{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\tilde{\mathbf{v}}^{n+1} - r\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right) + \boldsymbol{\nabla} p^n = \mathbf{f}^{n+1}$$

$$\tilde{p}^{n+1} = p^n - r\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right)$$

$$\varepsilon\left(\frac{\hat{\mathbf{v}}^{n+1}}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\hat{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\hat{\mathbf{v}}^{n+1}\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \hat{\mathbf{v}}^{n+1}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right)$$

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}}^{n+1} + \hat{\mathbf{v}}^{n+1}, \quad \text{and} \quad p^{n+1} = \tilde{p}^{n+1} - \frac{1}{\varepsilon}\left(\boldsymbol{\nabla}\cdot \mathbf{v}^{n+1} - q^{n+1}\right)$$

with $\tilde{\mathbf{v}}^{n+1}_{|\Gamma} = \mathbf{v}^{n+1}_D$ and $\hat{\mathbf{v}}^{n+1}_{|\Gamma} = 0$. Notice that (VPP$_{0,\varepsilon}$) is different from (VPP$_\varepsilon$) unless $\delta t = 1$. Indeed, summing now the two steps gives the discrete problem below which provides an additional dissipation of the velocity divergence for $\delta t < 1$:

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \boldsymbol{\nabla})\mathbf{v}^{n+1} - \frac{1}{\mathrm{Re}}\boldsymbol{\Delta}\mathbf{v}^{n+1} + \boldsymbol{\nabla} p^{n+1} = \mathbf{f}^{n+1} \qquad [9]$$

$$\varepsilon\,\frac{p^{n+1} - p^n}{\delta t} + \frac{1}{\delta t}\left(\boldsymbol{\nabla}\cdot \mathbf{v}^{n+1} - q^{n+1}\right) + \frac{r\varepsilon}{\delta t}\left(\boldsymbol{\nabla}\cdot \tilde{\mathbf{v}}^{n+1} - q^{n+1}\right) = 0. \qquad [10]$$

This results in a well-posed generalization of the method early proposed in [CAL 99] considering a singular vector correction step, with $\varepsilon = 0$ in fact, which has a unique solution only with an additional constraint for $\hat{\mathbf{v}}$, such that e.g.: $\boldsymbol{\nabla} \times \hat{\mathbf{v}} = 0$.

Defining $\mathbf{e}^n = \mathbf{v}^n - \mathbf{v}(t_n)$ and $\pi^n = p^n - p(t_n)$ as the whole velocity and pressure errors with the Navier-Stokes true solution, respectively, we prove in [ANG 08] the following result of first-order convergence in time for the time semi-discrete setting.

THEOREM 2.1 (ERROR ESTIMATES FOR (VPP$_\varepsilon$ – VPP$_{r,\varepsilon}$) WITH $\mathbf{v}_D = 0$ AND $q = 0$.) Assume $(\mathbf{v}, p)$ the solution of [1] smooth enough in time and space, well-prepared initial conditions and for (VPP$_{r,\varepsilon}$), small enough parameters such that $4r(\mathrm{Re}+\varepsilon) \leq 1$ and $4c(\Omega)\sqrt{\mathrm{Re}}\, r\varepsilon \leq \sqrt{\delta t}$, $c(\Omega)$ being the Poincaré constant, then there exists $C = C(\Omega, T, \mathrm{Re}, \mathbf{f}, \mathbf{v}_0, \mathbf{e}^0, \pi^0) > 0$ such that we have for all $n \in \mathbb{N}$ with $(n+1)\delta t \leq T$,

$$(i\text{-VPP}_{r,\varepsilon}) \qquad ||\mathbf{e}^{n+1}||_0^2 + \varepsilon\delta t\, ||\pi^{n+1}||_0^2 + \sum_{k=0}^{n} \frac{\delta t}{\mathrm{Re}} ||\boldsymbol{\nabla}\mathbf{e}^{k+1}||_0^2 \leq C\left(\delta t^2 + \varepsilon^2 \delta t^{\frac{3}{2}}\right),$$

$$(ii\text{-VPP}_{r,\varepsilon}) \qquad \sum_{k=0}^{n} \delta t\, ||\pi^{k+1}||_0^2 \leq C\left(\delta t^2 + \varepsilon^2\delta t\right), \; ||\boldsymbol{\nabla}\mathbf{e}^{n+1}||_0^2 \leq C\,\mathrm{Re}^2\left(\delta t + \varepsilon^2\right),$$

$$(iii\text{-VPP}_{r,\varepsilon}) \qquad \sum_{k=0}^{n} \delta t\, ||\boldsymbol{\nabla}\cdot \mathbf{v}^{k+1}||_0^2 = \sum_{k=0}^{n} \delta t\, ||\boldsymbol{\nabla}\cdot \mathbf{e}^{k+1}||_0^2 \leq C\left(\delta t + \varepsilon\right)\varepsilon\delta t^2.$$

$$(iii\text{-VPP}_{\varepsilon}) \qquad \sum_{k=0}^{n} \delta t\, ||\boldsymbol{\nabla}\cdot \mathbf{v}^{k+1}||_0^2 = \sum_{k=0}^{n} \delta t\, ||\boldsymbol{\nabla}\cdot \mathbf{e}^{k+1}||_0^2 \leq C\left(1 + \frac{\varepsilon^2}{\delta t^{\frac{5}{2}}} + \frac{\varepsilon}{\delta t}\right)\varepsilon\delta t.$$

3. Generalizations of (VPP$_{r,\varepsilon}$) methods for outflow BC or variable density flows

It is straightforward to write the above (VPP) methods with a higher-order scheme, e.g. the usual second-order Gear scheme (BDF2) as in [KHA 00, JOB 06].

1) **(VPP$_{r,\varepsilon}$) methods for incompressible and variable density flows**
By omitting here the convection term and with $q = 0$ for sake of shortness, the vector prediction and correction steps in section 2.2 are naturally modified as follows:

$$\frac{\varrho^{n+1} - \varrho^n}{\delta t} + \boldsymbol{\nabla}\cdot\left(\varrho^{n+1}\mathbf{v}^n\right) = 0$$

$$\varrho^{n+1}\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} - \boldsymbol{\nabla}\cdot \mu^{n+1}\left(\boldsymbol{\nabla}\tilde{\mathbf{v}}^{n+1} + (\boldsymbol{\nabla}\tilde{\mathbf{v}}^{n+1})^T\right) - r\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\tilde{\mathbf{v}}^{n+1}\right) + \boldsymbol{\nabla}p^n = \mathbf{f}^{n+1}$$

$$\varepsilon\left(\varrho^{n+1}\frac{\hat{\mathbf{v}}^{n+1}}{\delta t} - \boldsymbol{\nabla}\cdot \mu^{n+1}\left(\boldsymbol{\nabla}\hat{\mathbf{v}}^{n+1} + (\boldsymbol{\nabla}\hat{\mathbf{v}}^{n+1})^T\right)\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\hat{\mathbf{v}}^{n+1}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\tilde{\mathbf{v}}^{n+1}\right).$$

It is thus clear that the velocity correction $\hat{\mathbf{v}}$ becomes quasi-independent on the density ϱ and viscosity μ as $\varepsilon \to 0$, see section 4, which is not the case for scalar projection methods. These terms can be then dropped in practical algorithms for small enough ε.

2) **(VPP$_{r,\varepsilon}$) methods for open boundary conditions on a part Γ_N of Γ**
For a given stress vector $(\boldsymbol{\sigma}(\mathbf{v}, p)\cdot \mathbf{n})_{|\Gamma_N} \equiv -p\,\mathbf{n} + \mu\left(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T\right)\cdot\mathbf{n} = \mathbf{g}$, we now get to satisfy both the Neumann and Dirichlet velocity boundary conditions:

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}_D^{n+1} \text{ on } \Gamma_D, \quad -p^n\,\mathbf{n} + \mu^{n+1}\left(\boldsymbol{\nabla}\tilde{\mathbf{v}}^{n+1} + (\boldsymbol{\nabla}\tilde{\mathbf{v}}^{n+1})^T\right)\cdot\mathbf{n} = \mathbf{g}^{n+1} \text{ on } \Gamma_N$$

$$\hat{\mathbf{v}}^{n+1} = 0 \text{ on } \Gamma_D, \quad -(\tilde{p}^{n+1} - p^n)\,\mathbf{n} + \mu^{n+1}\left(\boldsymbol{\nabla}\hat{\mathbf{v}}^{n+1} + (\boldsymbol{\nabla}\hat{\mathbf{v}}^{n+1})^T\right)\cdot\mathbf{n} = 0 \text{ on } \Gamma_N.$$

4. Numerical experiments with the finite volume MAC scheme

The (VPP$_{r,\varepsilon}$) methods are implemented with a Navier-Stokes finite volumes solver on the staggered MAC mesh of size h issued from previous works; see [KHA 00]. The first test case is the unsteady Green-Taylor vortex such that the mean steady velocity field is of order 1 at $\mathrm{Re}=100$ like in [CAL 99]. The scheme is $\mathcal{O}(\delta t)$ in time for velocity and pressure with $r \geq 10^{-3}$, i.e. for $r\mathrm{Re} = \mathcal{O}(1)$, see Figure 1, whereas it is also $\mathcal{O}(h^2)$ in space as in [CAL 99]. Moreover, we find that the L^2-norm of the velocity divergence varies like $\mathcal{O}(\varepsilon\delta t)$, as expected by the theory; see Theorem 2.1. However, the pressure does not seem to converge for $r < 10^{-3}$, unless performing inner iterations as for the augmented Lagrangian which is not our goal. This is somewhat deceptive since we should expect convergence for smaller values of r in order to degrade the conditioning of the prediction step at the very least. The second benchmark problem is the Rayleigh-Bénard natural convection inside a square differentially heated vertical cavity at $Ra = 10^5$, the vertical walls being isothermal and the horizontal walls insulating. Here, we study the convergence properties of the penalty-correction step for this sharp test case at $t = 2\delta t$ with $\delta t = 1$. Again, we get the convergence of the velocity divergence as $\mathcal{O}(\varepsilon\delta t)$, whatever the viscosity term included in the penalty-correction step and also for $\mu = 0$. We can reach the machine precision of 10^{-15} for double precision floating point computations. Besides, the solution of the *penalty-correction step* proves to be all the cheaper as $\varepsilon\delta t$ tends to zero: typically one or two iterations of a preconditioned gradient algorithm, e.g. the ILU-BiCGStab solver, are sufficient for $\varepsilon\delta t \leq 10^{-6}$ and the condition number of the operator (preconditioned or not) also becomes more and more quasi-independent on the spatial mesh size h. Hence, the central issue for the proposed scheme is that the linear system associated with the projection step can be solved very easily because of the specific form of the right-hand side. Indeed, this latter lies more and more in the image of the left-hand side operator as ε is taken smaller and smaller; see the projection step in section 2.2.

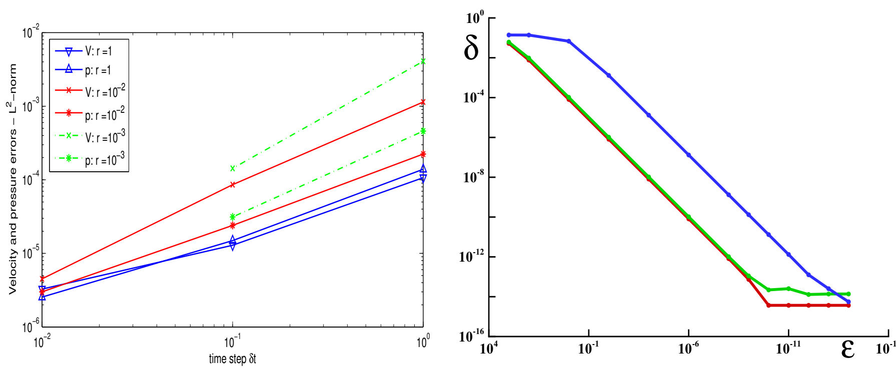

Figure 1. LEFT: *(VPP$_{r,\varepsilon}$) velocity and pressure convergence in time for the Green-Taylor vortex at* $\mathrm{Re}=100$, $t=10$ - $h=1/128$, $\varepsilon=10^{-4}$ *with* $||\boldsymbol{\nabla}\cdot \mathbf{v}^n||_{L^2} = \mathcal{O}(\varepsilon)$. RIGHT: *Convergence of the velocity correction step - divergence L^2-norm δ as $\mathcal{O}(\varepsilon)$ - for the natural convection at $Ra=10^5$ and $t=2\delta t$ with $\delta t=1$, $h=1/256$ – $\mu=0$ or* $1.85\,10^{-5}$ *(idem) and* $\mu=1.85\,10^{-1}$.

5. Perspectives

The generalization of the (VPP$_{r,\varepsilon}$) methods for low Mach number dilatable flows in the same spirit of what was done in [JOB 07] is the subject of a future work. Moreover, their extension to more general vector problems for fictitious domain methods with immersed moving and/or deformable interfaces, e.g. multiphase flows, particulate flows or fluid-structure interaction problems, is an undergoing project.

6. References

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